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An extended Galilean group and its application to time operators

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Abstract. The representation theory of the Galilean group extended by a one-dimensional group of dilations (which leave invariant the free Schrödinger equation) is studied. The existence of position and time operators is investigated from the standpoint of the imprimitivity theorem, for quantum particles whose states carry a projective representation of the extended group. Position and time operators are shown to exist for some two-component direct sums with non-trivial multipliers and for vector representations with zero helicity.

1. Introduction

The existence of commuting position operators for quantum particles was seen by Mackey (1963), and Wightman (1962) to be a question that could be settled using the imprimitivity theorem (Mackey 1949). Their method is to consider the representation L of the three-dimensional Euclidean group $\mathscr{E}(3)$ resulting from the restriction of the particle's projective representation of the whole kinematical group. If a system of imprimitivity P for the usual action of $\mathscr{E}(3)$ on $\mathbb{R}^3$ can be defined for L then the corresponding particle is said to be localizable in space. Three position operators may be obtained from the projection valued measure whenever it exists.

The idea that time should be treated as an observable in quantum theory is not new and has been considered recently from various points of view (Rosenbaum 1969, Almond 1973, Olkhovsky *et al* 1974). In this paper we present a treatment of position and time operators for free non-relativistic quantum systems based on the imprimitivity theorem.

Following Almond (1973) we choose for our kinematical group the Galilean transformations of space–time points (x, t) enlarged by the one-dimensional group of dilations $d : (x, t) \rightarrow (dx, d^2t)$, where d is real and positive. We denote the enlarged group by $\mathscr{D}$. Let $\mathbb{R}_T$ be the subgroup of time translations of $\mathscr{D}$ and suppose U is a projective unitary representation of $\mathscr{D}$. We say that the particle corresponding to U is localizable in space and time if there exists a system of imprimitivity P, based on $\mathbb{R}^4$ for the restriction of U to the direct product $\mathscr{E}(3) \times \mathbb{R}_T$. Such a P gives four operators, three are interpreted as position operators, the fourth as time.

In § 2 we study the representation theory of $\mathscr{D}$ and find that three distinct classes, denoted $\boldsymbol{U}$, $\boldsymbol{V}^+$ and $\boldsymbol{V}^-$ of irreducible projective representation occur for each non-trivial multiplier. Those of class $\boldsymbol{U}$ remain irreducible on the Galilean subgroup. For trivial multipliers we find four classes of unitary irreducible representations labelled $\boldsymbol{Q}$, $\boldsymbol{R}^+$, $\boldsymbol{R}^-$ and $\boldsymbol{S}$. In § 3 we construct the systems of imprimitivity required in § 4 for the localizability considerations. We find that none of the class $\boldsymbol{U}$ irreducibles are localizable and that the simplest localizable representation built from the classes $\boldsymbol{V}^+$ and $\boldsymbol{V}^-$ is a direct sum of two irreducibles. For trivial multipliers only a subset of the $\boldsymbol{Q}$ are seen to be localizable.

Throughout the remaining sections we use the word representation to mean 'unitary continuous representation' and replace the group SO(3) by its cover SU(2) in the usual definitions of the Euclidean and Galilean groups.

2. The representations of $\mathscr{D}$

2.1. Notation

If we use the notation $\mathbb{R}^3_S$ for the space translation subgroup of $\mathscr{D}$ and $\mathbb{R}^+$ for the multiplicative group of positive reals then each element g of $\mathscr{D}$ may be written as (h, b, u, v, d), for $h \in SU(2)$, $b \in \mathbb{R}_T$, $u \in \mathbb{R}^3_S$, $v \in \mathbb{R}^3$ and $d \in \mathbb{R}^+$. g maps the point (x, t) to (x', t') where,

$$x' = d^2(hx + vt + u) \tag{1}$$

$$t' = d(t + b). \tag{2}$$

The notation hx means $\delta(h)x$, where δ is the covering homomorphism $\delta : SU(2) \to SO(3)$. From this we obtain the product law

$$(h', b', u', v', d')(h, b, u, v, d) = (h'h, b + (b'/d^2), h'u + dbv' + (u'/d), h'v + dv', d'd)$$

for $\mathscr{D}$. The Galilean subgroup is the set of elements for which $d = 1$. Let $H^2(\mathscr{D}, U(1))$ denote the group of $U(1)$ multipliers for $\mathscr{D}$. Almond (1973) has shown that $H^2(\mathscr{D}, U(1))$ is isomorphic to $\mathbb{R}$ and that each $\sigma \in H^2(\mathscr{D}, U(1))$ is equivalent to one of the form

$$\sigma_m(g', g) = \exp[im(\tfrac{1}{2}d^2bv'^2 + dh'u \,.\, v')]$$

where $g' = (h', b', u', v', d')$ and m is a real parameter that changes the equivalence class in $H^2(\mathscr{D}, U(1))$. Corresponding to each non-trivial $\sigma \in H^2(\mathscr{D}, U(1))$ there is a non-trivial central extension $\mathscr{D}_\sigma$ of $\mathscr{D}$ by $U(1)$ (Varadarajan 1970). The multiplication in $\mathscr{D}_\sigma$ is defined by $(g', z')(g, z) = (g'g, \sigma(g', g)z'z)$, for $z', z \in U(1)$. We call a projective representation corresponding to a non-trivial σ a σ-representation and adopt the notation $\hat{\mathscr{A}}^\sigma$ for the set of all irreducible σ-representations of a group $\mathscr{A}$. $\hat{\mathscr{A}}$ denotes the vector representations. To construct the $\hat{\mathscr{D}}^\sigma$ we use the well known device of first constructing the $\hat{\mathscr{D}}_\sigma$.

2.2. The projective representations

In each extension $\mathscr{D}_\sigma$ the direct product $\mathbb{R}^3_S \times \mathbb{R}_T \times U(1)$ is normal and each $\mathscr{D}_\sigma$ is isomorphic to a semi-direct product $(\mathbb{R}^3_S \times \mathbb{R}_T \times U(1))$ ⓐ $\mathscr{K}$ where $\mathscr{K}$ is the subgroup of elements $(h, 0, 0, v, d)$. The homomorphism $a : \mathscr{K} \to \operatorname{aut}(\mathbb{R}^3_S \times \mathbb{R}_T \times U(1))$ depends on σ and is described by,

$$b' = d^2 b$$

$$u' = d(hu + bv)$$

$$z' = \exp[im(\tfrac{1}{2}bv^2 + hu \,.\, v)]z$$

when σ is equivalent to σ_m. Computing the dual action of a we find,

$$E' = d^{-2}(E - v \,.\, hp + \tfrac{1}{2}nmv^2) \tag{3}$$

$$p' = d^{-1}(hp - nmv) \tag{4}$$

$$n' = n$$

for $p \in \hat{\mathbb{R}}^3_S$, $E \in \hat{\mathbb{R}}_T$ and n integral. To construct $\hat{\mathscr{D}}_\sigma$ it is necessary only to consider the orbits through $n = -1$. We observe that to each value of n there are three distinct orbits of $\mathscr{K}$ in $\hat{\mathbb{R}}^3_S \times \hat{\mathbb{R}}_T$, those through $n = -1$ are labelled by X^0, X^+ and X^- corresponding respectively to the choices $(0, 0, -1)$, $(0, 1, -1)$ and $(0, -1, -1)$ of initial point. The stability groups are $SU(2) \times \mathbb{R}^+$, $SU(2)$ and $SU(2)$. From the relationship $p'^2 + 2mE' = d^{-2}(p^2 + 2mE)$ we deduce that X^0 is the parabola $p^2 + 2mE = 0$, which is also an orbit for the Galilean subgroup. X^+ and X^- are respectively the unions

$$\bigcup_{\varepsilon>0} X^\varepsilon \quad \text{and} \quad \bigcup_{\varepsilon<0} X^\varepsilon,$$

where X^ε is the Galilean orbit $p^2 + 2mE = \varepsilon$. Mackey's method enables us to label the irreducible projective representations of $\mathscr{D}$ associated with the orbits X^0, X^+ and X^- as $U^{m,j,\alpha}$, $V^{m,j,+}$ and $V^{m,j,-}$ respectively. Here j refers to the $(2j+1)$-dimensional member of $\widehat{SU}(2)$ and α to the element $\gamma_\alpha : x \to x^{i\alpha}$ of $\hat{\mathbb{R}}^+$, for $\alpha \in \mathbb{R}$. For our purposes it is convenient to have the Wigner function realizations (Niederer and O'Raifeartaigh 1974) of these representations. They are,

$$(U_g^{m,j,\alpha} f)(p) = d^{\frac{3}{2}+i\alpha} \exp\left[i\left(u \,.\, p - \frac{bp^2}{2m}\right)\right] D^j(h) f(h^{-1}(dp - mv))$$

for $f \in \mathscr{L}^2\,(\mathbb{R}^3, \mathbb{C}^{2j+1}, d^3p)$, and

$$(V_g^{m,j,\pm} f)(p, \varepsilon) = d^{5/2} \exp\left[i\left(\frac{b}{2m}(\varepsilon - p^2) + u \,.\, p\right)\right] D^j(h) f(h^{-1}(dp - mv), d^2\varepsilon)$$

for $f \in \mathscr{L}^2(\mathbb{R}^3 \times \mathbb{R}^\pm, \mathbb{C}^{2j+1}, d^3p\,d\varepsilon^\pm)$, where $\mathbb{R}^+$ and $\mathbb{R}^-$ denote the positive and negative real numbers and $d\varepsilon^\pm$ is the restriction of Lebesgue measure to $\mathbb{R}^\pm$. We observe that each U is irreducible on the Galilean subgroup.

2.3. The vector representations

$\mathscr{D}$ is a semi-direct product of $\mathbb{R}^3_S \times \mathbb{R}_T$ with the group $\mathscr{K}$. The dual action of $\mathscr{K}$ is obtained by setting $m = 0$ in equations (3) and (4). Four orbits occur, consequently we have four classes $\boldsymbol{Q}$, $\boldsymbol{R}^+$, $\boldsymbol{R}^-$, and $\boldsymbol{S}$ of irreducible representations. The stability groups are isomorphic respectively to the two-dimensional Euclidean group $\mathscr{E}(2)$, $\mathscr{E}(3)$, $\mathscr{E}(3)$ and $\mathscr{K}$. In terms of Wigner functions we have,

$$(Q_g f)(p, E) = d^{5/2} \exp[i(u \,.\, p + bE)] M_{(h_0, v_0)} f(dh^{-1}p, d^2E + dv \,.\, p)$$

for $f \in \mathscr{L}^2\,(\mathbb{R}^4, \mathscr{H}(M), d^3p\,dE)$. Here h_0 is the Wigner rotation, $v_0 \in \mathbb{R}^2$ and M is an irreducible representation of $\mathscr{E}(2)$ in the Hilbert space $\mathscr{H}(M)$.

$$(R_g^+ f)(E) = d \exp(ibE) L(h, d^{-1}|E|^{-1/2} v) f(d^2E)$$

for $f \in \mathscr{L}^2\,(\mathbb{R}^\pm, \mathscr{H}(L), dE^\pm)$ and $L \in \widehat{\mathscr{E}(3)}$ in $\mathscr{H}(L)$. Those belonging to class $\boldsymbol{S}$ are the irreducible representations of $\mathscr{K}$. $\mathscr{K}$ is itself a semi-direct product and Mackey's theory may again be used. We obtain,

$$(S^n_g f)(p) = d^{-3/2} \exp[\mathrm{i}(v\,.\,p + n\phi_0)] f(d^{-1}h^{-1}p)$$

for $f \in \mathscr{L}^2(\mathbb{R}^3, \mathbb{C}, d^3p)$, where ϕ_0 is the angle of h_0 and n is integral, and

$$S^{j,\alpha} = D^j \otimes \gamma_\alpha.$$

3. Systems of imprimitivity for $\mathscr{E}(3) \times \mathbb{R}_\mathrm{T}$

Restricting the action of $\mathscr{D}$ on (x, t) to $\mathscr{E}(3) \times \mathbb{R}_\mathrm{T}$ defines $\mathbb{R}^4$ as transitive $\mathscr{E}(3) \times \mathbb{R}_\mathrm{T}$ space. As a consequence the $\mathscr{E}(3) \times \mathbb{R}_\mathrm{T}$ systems of imprimitivity for this action may be written down, up to equivalence, using the imprimitivity theorem. In particular we have the following lemma.

Lemma 1. If P is a system of imprimitivity for the representation U of $\mathscr{E}(3) \times \mathbb{R}_\mathrm{T}$, based on $\mathbb{R}^4$ under the restriction of the action described by equations (1) and (2), then the pair (U, P) is unitarily equivalent to a pair (V, F) in $\mathscr{L}^2(\mathbb{R}^4, \mathscr{H}, d^3x dt)$ given by

$$(V_{(h,b,u)} f)(x, t) = D(h) f(h^{-1}(x-u), t-b)$$

$$(F_B f)(x, t) = \chi_B(x, t) f(x, t)$$

where D is a representation of SU(2) in $\mathscr{H}$ and B is a Borel set of $\mathbb{R}^4$ with characteristic function χ_B.

When D is equivalent to D^j the corresponding V of lemma 1 has the decomposition

$$\bigoplus_{n=-j}^{j} \int_{-\infty}^{\infty} \left(\int_0^{\infty} W^{n,r,E}\, \mathrm{d}r \right) \mathrm{d}E$$

where $W^{n,r,E} \in \widehat{\mathscr{E}(3)} \times \hat{\mathbb{R}}_\mathrm{T}$ is described in terms of complex valued functions on the sphere $p^2 = r$ by

$$(W^{n,r,E} f)(p) = \exp[\mathrm{i}(u\,.\,p + bE + n\phi_0)] f(h^{-1}p).$$

4. Localizability

4.1.

Restricting the $U^{m,j,\alpha}$ to $\mathscr{E}(3) \times \mathbb{R}_\mathrm{T}$ we obtain

$$(Wf)(p) = \exp\left[\mathrm{i}\left(u\,.\,p - \frac{bp^2}{2m}\right)\right] D^j(h) f(h^{-1}p)$$

in $\mathscr{L}^2(\mathbb{R}^3, \mathbb{C}^{2j+1}, d^3p)$, which is clearly not of the form required by lemma 1. We conclude that none of the particles corresponding to these representations are localizable in space and time.

4.2.

The $V^{m,j,\pm}$ restricted to $\mathscr{E}(3) \times \mathbb{R}_\mathrm{T}$ give

$$(Wf)(p, v) = \exp\left[\mathrm{i}\left(\frac{b}{2m}(v - p^2) + u\,.\,p\right)\right] D^j(h) f(h^{-1}p, v)$$

in $\mathscr{L}^2(\mathbb{R}^3\times\mathbb{R}^\pm, \mathbb{C}^{2j+1}, d^3pd\varepsilon^\pm)$. Decomposing we obtain

$$W^+ \cong \bigoplus_{n=-j}^{j} \int_0^\infty \left(\int_0^\infty W^{n,r,E}\, \mathrm{d}r\right) \mathrm{d}E$$

and

$$W^- \cong \bigoplus_{n=-j}^{j} \int_{-\infty}^0 \left(\int_0^\infty W^{n,r,E}\, \mathrm{d}r\right) \mathrm{d}E.$$

Neither of these is equivalent to any of the representations described in lemma 1. However, the direct sum $W^+\oplus W^-$ does have a system of imprimitivity so that the particle corresponding to the representation $V^{m,j,+}\oplus V^{m,j,-}$ is localizable in space and time. Its time operator is $-\mathrm{i}\partial/\partial E$. We note further that $V^{m,j,+}\oplus V^{m,j',-}$ cannot be localizable unless $j = j'$. The possibility of adding representations with different m is excluded by Bargmann's super-selection rule (Levy-Leblond 1963).

4.3.

On restricting the $\boldsymbol{Q}$ representation when M is $M^n: (h(\phi), y) \rightarrow \exp(\frac{1}{2}\mathrm{i}n\phi)$, where

$$h(\phi) = \begin{pmatrix} \exp(\frac{1}{2}\mathrm{i}\phi) & 0 \\ 0 & \exp(-\frac{1}{2}\mathrm{i}\phi) \end{pmatrix}$$

we obtain

$$(Wf)(p, E) = \exp[\mathrm{i}(u\,.\,p + bE + n\phi_0)]f(h^{-1}p, E)$$

in $\mathscr{L}^2(\mathbb{R}^4, \mathbb{C}, d^3pdE)$. If Q^n denotes the representation induced using M^n then by performing a Fourier transformation we observe that Q^0 is the only localizable irreducible but that any direct sum of the form $\oplus_{n=-q}^{q} Q^n$ is localizable.

When M is equivalent to

$$(M^{\rho,1}_{(h(\phi),\,y)}k)(\psi) = \exp\{\mathrm{i}\rho \operatorname{Re}[(y^1 + \mathrm{i}y^2)\exp(\mathrm{i}\psi)]\}k(\psi - \phi)$$

in $\mathscr{L}^2(U(1), \mathbb{C}, d\psi)$ the restriction to $\mathscr{E}(3)\times\mathbb{R}_\mathrm{T}$ is

$$(Wf)(p, E) = \exp[\mathrm{i}(u\,.\,p + bE)]M^{\rho,1}_{(h_0,\,0)}f(h^{-1}p, E).$$

By expanding f in the form

$$f(h^{-1}p, E)(\psi) = \sum_{n=-\infty}^{\infty} c_n(h^{-1}p, E)\exp(\mathrm{i}n\psi)$$

we see that $M^{\rho,1}_{(h_0,0)}$ is infinite diagonal matrix with entries $\exp(\mathrm{i}n\phi_0)$ for integral n. Hence the representation of $\mathscr{E}(3)\times\mathbb{R}_\mathrm{T}$ for this case is

$$\bigoplus_{n=-\infty}^{\infty} W^n$$

where

$$W^n = \int_{-\infty}^\infty \left(\int_0^\infty W^{n,r,E}\, \mathrm{d}r\right) \mathrm{d}E,$$

so that these do not correspond to localizable particles.

Finally if M is the representation

$$(M^{\rho,2}_{(h(\phi),\, y)}k)(\psi)=\exp\{\tfrac{1}{2}\mathrm{i}\phi+\mathrm{i}\rho\ \mathrm{Re}[(y^1+\mathrm{i}y^2)\exp(\mathrm{i}\psi)]\}k(\psi-\phi)$$

the factor $\exp(\frac{1}{2}\mathrm{i}\phi)$ has the effect of multiplying the matrix of the previous case by $\exp(\frac{1}{2}\mathrm{i}\phi_0)$ so that on $\mathscr{E}(3)\times\mathbb{R}_\mathrm{T}$ we have

$$\bigoplus_{n=\pm\frac{1}{2},\pm\frac{3}{2},\pm\frac{5}{2},\ldots} W^n$$

which is not localizable.

4.4.

Clearly the classes $\boldsymbol{R}^{\pm}$ and $\boldsymbol{S}$ are not localizable in the sense defined.

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